MA 222 - Analysis II: Measure and Integration (JAN-APR, 2016)

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Problem Set 3

1. Let μ_n be the L-measure on $\mathbb{R}^n, n > 1$.
   (a) Show that, $\mu_n(\mathbb{R}^k) = 0$ for all $1 \leq k \leq n-1$.
   (b) Let $A = \{(x,y) \in \mathbb{R}^2 : xy = 1\}$. Show that $\mu_2(A) = 0$.
   (c) Prove $\mu_3(S^2) = 0$, where S^2 in the unit sphere in $\mathbb{R}^3$.

2. (a) Let $f : [0,\infty) \longrightarrow \mathbb{R}$ be uniformly continuous and integrable (i.e. $\int_0^\infty |f| < \infty$). Show that $\lim\limits_{x\to\infty} f(x) = 0$.
   (b) Show, by examples, that the result need not hold if we drop any one of the assumption.

3. Check for L-integrability and find the value of the integral whenever it is possible.
   (a) $f(x) = \frac{1}{x^\alpha}$ on $(0,1), \alpha \in \mathbb{R}$
   (b) $f(x) = \exp(-x)$ on $[0,\infty)$
   (c) $f(x) = \exp(x)$ on $[0,\infty)$
   (d) $f(x) = \frac{1}{x}\sin(\frac{1}{x})$
   (e) $f(x) = \frac{x^{(n-1)}}{(1+x^2)^k}$ on $(0,\infty)$

4. Let f be R-integrable and g be L-integrable on $(0,1)$. Further, $\displaystyle\int_0^1 |f-g| = 0$. Is g in R-integrable ?

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative function. Define $\nu : \mathcal{M} \longrightarrow \mathbb{R}$, where $\mathcal{M}$ is the Lebesgue σ-algebra of measurable functions, by $\nu(E) = \displaystyle\int_E f$. Prove that ν is countably additive; that is if E_i's are disjoint measurable sets, then $\nu(\bigcup_{i=1}^\infty E_i) = \sum_{i=1}^\infty \nu(E_i)$. In other words, $\displaystyle\int_{\bigcup_{i=1}^\infty E_i} f = \sum_{i=1}^\infty \int_{E_i} f$.

6. (a) State and prove the generalized version of LDC.
   (b) Give an example to show that LMC will not hold if the sequence is decreasing.
   (c) Give an example to show that the strict inequality can hold in Fatou's lemma.
   (d) Prove Fatou's lemma using bounded convergence theorem.
   (e) Derive MCT from Fatou's lemma.

7. Let $f \geq 0$ be measurable and $\displaystyle\int f = 0$, then show that $f = 0$ a.e.

8. Let E be measurable. Show that

$$\lim_{\delta \to 0} \frac{\mu(E \bigcap (x - \delta, x + \delta))}{2\delta}$$

   exists a.e. and equal to $\chi_E(x)$ a.e. (Hint: Use regularity for E and then Urysohn's lemma).

9. Let μ_1, μ_2 be L-measures on $\mathbb{R}^1, \mathbb{R}^2$ respectively.

   (a) Let E be a measurable subset of $\mathbb{R}$ and let $\sigma(E) = \{(x, y) \in \mathbb{R}^2 : x - y \in E\}$. Show that $\sigma(E)$ is μ_2-measurable.
   (b) Let f be μ_1-measurable on $\mathbb{R}$. Define $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $F(x, y) = f(x - y)$ S.T. F is μ_2-measurable.
   (c) Let f, g be μ_1-measurable. Show that the product $\phi(x, y) = f(x - y)g(y)$ is μ_2-measurable.